Let us sumarize what we obtained in the previous lecture:

Let $\pi: X \to Y$ be an abelian cover with group G. Then G acts on $\pi_* \mathcal{O}_X$ and it splits $\pi_* \mathcal{O}_X$ as direct sum of isotypic components V_χ of character χ:

$$\pi_* \mathcal{O}_X = \bigoplus_{\chi \in G^*} V_\chi$$

V_χ is an invertible sheaf:

given $q \in Y$ and a fundamental neigh. V of q for π, then

$$\tau_V^\chi := \begin{cases} \sum_{g \in G} \overline{\chi(g)} \cdot g^* \mathbb{1}_\mathcal{U} & \text{if } q \notin \mathrm{supp}(D) \\ \sum_{g \in G/\langle h \rangle} \overline{\chi(g)} \cdot g^*(t^{r_\chi^h} \mathbb{1}_\mathcal{U}) & \text{if } q \in \mathrm{supp}(\Delta_h) \setminus \mathrm{Sing}(D) \end{cases}$$

where t is the local parameter of a irreducible component $T \subseteq X$, $T \cap \mathcal{U} = (t=0)$, lying on $\Delta_h \cap V$, and $0 \le r_\chi^h \le |h|-1$ is the unique integer s.t. $\chi(h) = e^{\frac{2\pi i}{|h|} \cdot r_\chi^h}$.

Then $\mathcal{O}_{Y|V} \xrightarrow{\sim} V_{\chi|V}$ is an iso of sheaves.

$$\alpha \longmapsto \alpha \cdot \tau_V^\chi$$

Remark 1 By def. of τ_V^χ, then

$$\tau_{V|\mathcal{U}}^\chi = \begin{cases} \mathbb{1}_\mathcal{U} & \text{if } q \notin \mathrm{supp}(D) \\ t^{r_\chi^h} \cdot \mathbb{1}_\mathcal{U} & \text{if } q \in \mathrm{supp}(\Delta_h) \setminus \mathrm{Sing}(D) \end{cases}$$

Similarly, given $g \in G/\langle h \rangle$, then

$$\tau^{\chi}_{V|_{g\cdot U}} = \begin{cases} \overline{\chi(g)} \cdot 1\!\!1_{g\cdot U} & \text{if } q \notin supp(D) \\ \overline{\chi(g)} \cdot (g^* t)^{r_\chi^h} \cdot 1\!\!1_{gU} & \text{if } q \in supp(\Delta_h) \setminus Sing(D) \end{cases}$$

Def $\mathcal{L}_\chi := V_\chi^{-1}$, so we have $\pi_* \mathcal{O}_X = \bigoplus_{\chi \in G^*} \mathcal{L}_\chi^{-1}$.

$\{D_g\}_{g \in G}$ and $\{\mathcal{L}_\chi\}_{\chi \in G^*}$ are called Building Data of $\pi : X \to Y$.

Remark 3 The cocycles of $\mathcal{L}_\chi = V_\chi^{-1}$ are

$$g_{V_2 V_1} = \frac{\tau^{\chi}_{V_2}}{\tau^{\chi}_{V_1}},$$

so a global (holom.) section of $\pi^* \mathcal{L}_\chi$ is

$$s_\chi = \{ (\pi^{-1}(V), \tau^{\chi}_{V}) \}_{V \subseteq Y \text{ fund. neigh.}}$$

We can state and prove

§9. Pardini Existence Theorem

Let Y be a smooth algebraic variety and let $\pi : X \to Y$ be an abelian cover of Y with Galois group G, X normal and building data $\{L_\chi\}_{\chi\in G^*}$, $\{D_g\}_g$.
Then, for any pairs of characters $\chi, \eta \in G^*$

$$(*) \qquad L_\chi + L_\eta = L_{\chi\eta} + \sum_{g\in G} \left\lfloor \frac{r_\chi^g + r_\eta^g}{|g|} \right\rfloor D_g$$

Conversely, given

- a collection of line bundles $\{L_\chi\}_{\chi\in G^*}$ of Y labeled by the characters of G,
- a collection of effective DIVISORS $\{D_g\}_{g\in G}$ indexed by the elements of G;

with the property that the linear equations $(*)$ hold for any pair $\chi, \eta \in G^*$, then it there exists an abelian cover $\pi : X \to Y$ with Galois group G.

If the cover so constructed is normal, then it has building data $\{L_\chi\}_{\chi\in G^*}$ and $\{D_g\}_{g\in G}$.

If Y is complete, then the building data determine the cover up to isomorph. of Galois covers.

Proof Let us consider the sections s_χ, $\chi \in G^*$ of $\pi^* L_\chi$ defined above. We observe that a global section of $\pi^*(L_\chi \otimes L_\eta \otimes L_{\chi\eta}^{-1})$ on X is $\frac{s_\chi s_\eta}{s_{\chi\eta}}$, which is a G-invariant section by construction. Thus $\frac{s_\chi s_\eta}{s_{\chi\eta}}$ lives on Y, and it is a section of $L_\chi \otimes L_\eta \otimes L_{\chi\eta}^{-1}$, still by construction.

Let $V \subseteq Y$ fund. open neigh. of Y for π.

If V does not intersect any $\Delta_h \subseteq D$, $h \in G$, then $\pi^{-1}(V) = \bigsqcup_{g \in G} g \cdot \mathcal{U}$, $\mathcal{U} \xrightarrow{\pi|_\mathcal{U}} V$ is an iso, and

$\tau_V^\chi = \sum_{g \in G} \overline{\chi(g)} \cdot g^* \mathbb{1}_\mathcal{U}$. But then

$$\frac{s_\chi s_\eta}{s_{\chi\eta}}\Big|_\mathcal{U} = \frac{\mathbb{1}_\mathcal{U} \cdot \mathbb{1}_\mathcal{U}}{\mathbb{1}_\mathcal{U}} = \mathbb{1}_\mathcal{U}, \quad \frac{s_\chi s_\eta}{s_{\chi\eta}}\Big|_{g\mathcal{U}} = \frac{\overline{\chi(g)}\,\overline{\eta(g)}\,\mathbb{1}_{g\mathcal{U}}}{\overline{\chi\eta(g)}\,\mathbb{1}_{g\mathcal{U}}} = \mathbb{1}_{g\mathcal{U}}$$

$$\Rightarrow \frac{s_\chi s_\eta}{s_{\chi\eta}}\Big|_{\pi^{-1}(V)} = 1 \quad \Rightarrow \frac{s_\chi s_\eta}{s_{\chi\eta}} \text{ does } \underline{\text{not}} \text{ vanish on } V.$$

Instead, assume $V \cap \Delta_h \neq \emptyset$ for some $\Delta_h \subseteq D$, $h \in G$. Then, up to restrict V, we can say that the local coordinates on V are $(y_1, y_2, \dots y_n)$, $\Delta_h = (y_1 = 0)$, that $\pi^{-1}(V) = \bigsqcup_{g \in G/\langle h \rangle} g \cdot \mathcal{U}$, with $h \cdot \mathcal{U} = \mathcal{U}$ and $\mathcal{U}/\langle h \rangle \cong V$, and given the irreducible component $T := \pi^{-1}(\Delta_h) \cap \mathcal{U}$, then $T = (t = 0)$, $\mathcal{U}$ has local coordinates $(t, z_2, \dots z_n)$, $\langle h \rangle$ acts locally as $(t, z_2, \dots z_n) \xmapsto{h} (\zeta t, z_2, \dots z_n)$, and finally

$(t, z_2, \dots z_n) \xrightarrow{\pi} (t^{|h|}, z_2, \dots z_n)$ (so $y_1 = t^{|h|}$).

In this case, we know

$$\tau_V^\chi := \sum_{g \in G/\langle h \rangle} \overline{\chi(g)}\, g^*\left(t^{r_\chi^h} \mathbb{1}_U\right) \quad \text{on } \pi^{-1}(V), \text{ and}$$

$$\tau^\chi_{V|U} = t^{r_\chi^h} \cdot \mathbb{1}_U \quad \text{on } U$$

Thus, $\dfrac{s_\chi s_\eta}{s_{\chi\eta}}\Big|_U = \dfrac{\tau_V^\chi \tau_V^\eta}{\tau_V^{\chi\eta}} = \dfrac{t^{r_\chi^h} \cdot t^{r_\eta^h}}{t_V^{r_{\chi\eta}^h}} \mathbb{1}_U = t^{r_\chi^h + r_\eta^h - r_{\chi\eta}^h} \cdot \mathbb{1}_U$

Instead, on gU, $g \in G/\langle h \rangle$, we have

$$\tau^\chi_{V|gU} = \overline{\chi}(g) \cdot (g^* t)^{r_\chi^h} \cdot \mathbb{1}_{gU}$$

$$\Rightarrow \frac{s_\chi s_\eta}{s_{\chi\eta}}\Big|_{gU} = (g^*t)^{r_\chi^h + r_\eta^h - r_{\chi\eta}^h} \mathbb{1}_{gU}$$

Then $\dfrac{s_\chi s_\eta}{s_{\chi\eta}}\Big|_{\pi^{-1}(V)} = \displaystyle\sum_{g \in G/\langle h \rangle} (g^*t)^{r_\chi^h + r_\eta^h - r_{\chi\eta}^h} \cdot \mathbb{1}_{gU}$

By definition of r_χ^h, then

$$r_\chi^h + r_\eta^h - r_{\chi\eta}^h = \begin{cases} |h| & \text{if } r_\chi^h + r_\eta^h \geq |h| \\ 0 & \text{otherwise} \end{cases}$$

If it is zero, then $\dfrac{s_\chi s_\eta}{s_{\chi\eta}}\Big|_{\pi^{-1}(V)} = 1 \Rightarrow \dfrac{s_\chi s_\eta}{s_{\chi\eta}}$ does not vanish on V.

Instead, if it is $|h|$, we have

$$\frac{s_\chi s_\eta}{s_{\chi\eta}}\Big|_{\pi^{-1}(V)} = \sum_{g \in G/\langle h \rangle} (g^*t)^{|h|} \cdot \mathbb{1}_{gU} = \sum_{g \in G/\langle h \rangle} g^*\left(t^{|h|}\right) \mathbb{1}_{gU}$$

↑ G-invariant function living on V.

However, $y_1 = t^{|h|}$ by construction of V, so

$$\frac{s_x s_\eta}{s_{x\eta}}|_V = y_1$$ on V, and $y_1 = 0$ is

the zero locus of Δ_h on V. Thus

$$\mathrm{div}\left(\frac{s_x s_\eta}{s_{x\eta}}\right)|_V = \Delta_h$$

This holds for any irred. component Δ_h of D_h

Thus, all the divisors D_h, with the property that $r_x^h + r_\eta^h - r_{x+\eta}^h = |h|$ (namely $\left\lfloor \frac{r_x^h + r_\eta^h}{|h|} \right\rfloor = 1$) occurs on $\mathrm{div}\left(\frac{s_x s_\eta}{s_{x\eta}}\right)$, and no others divisors occur on it. This proves $(*)$.

Let us assume now to have a set of line bundles $\{\mathcal{L}_x\}_{x \in G^*}$ and divisors $\{D_g\}_{g \in G}$ of Y for which equations $(*)$ hold.

We consider the vector bundle $V(\bigoplus_{x \neq 1_G} \mathcal{L}_x) \xrightarrow{\pi} Y$.

It is always possible to choose a open cover of Y, $\{V\}_{V \subseteq Y}$ such that any V trivializes simultaneously $\mathcal{L}_x$, $V(\mathcal{L}_x)(\pi^{-1}(V)) \longrightarrow Y \times \mathbb{C}$

$$p \longmapsto (\pi(p), u_x(p))$$

Thus we have local coordinates $(u_\chi : \chi \in G^* \setminus \{1_G\})$ trivializing $V(\bigoplus_{\chi \neq 1_G} L_\chi)$ on $\pi^{-1}(V)$.

For any $g \in G$, we define the action on $V(\bigoplus_{\chi \neq 1_G} L_\chi)$:

$$(u_\chi : \chi \in G^* \setminus \{1_G\}) \overset{g}{\longmapsto} (\chi(g) u_\chi : \chi \in G^* \setminus \{1_G\})$$

The local action is compatible with the change of the chart, so it extends to the entire $V(\bigoplus_{\chi \neq 1_G} L_\chi)$. For any $g \in G$, let us choose $\sigma_g \in H^0(Y, \mathcal{O}_Y(D_g))$ with $\operatorname{div}(\sigma_g) = D_g$.

Finally, we define X on the local chart $\pi^{-1}(V)$:

$$X \cap \pi^{-1}(V) := \left\{ u_\chi \cdot u_\eta = \left(\prod_{r_\chi^h + r_\eta^h \geq |h|} \sigma_h \right) \cdot u_{\chi\eta} \right\}$$

We notice that from $(*)$, then we can glue these sets and obtain $X := \bigcup_{V \subseteq Y} X \cap \pi^{-1}(V)$.

By construction of X, then the action of G on $V(\bigoplus_{\chi \neq 1_G} L_\chi)$ extends to an action of X.

Thus, we have a Galois covering $\pi|_X : X \to Y$, $(q, u_\chi : \chi \in G \setminus 1_G) \mapsto q$ of Y with group G.

Assume X is normal, so the theory of norm. ab. covers holds for $\pi_{|X}: X \to Y$, which has then some building data. The ram. locus of $\pi_{|X}$ consists of those points with no triv. stab. It is easy to see from equations ☺ that $p \in X$ has no trivial stab $\Leftrightarrow$ $p \in supp(div(\sigma_h))$ for some σ_h. Thus, $D = \sum_{h \in G}' supp(div(\sigma_h)) = \sum_{h \in G}' red(D_h)$ where $red(D_h)$ is the reduced divisor of D_h. However, X normal forces D_h to be already reduced, and D_g and D_h doesn't have common components. (we will see this in the next lectures when we will study the normality of a standard abelian cover).

This means $D = \sum_{h \in G}' D_h$.

To the other side,

$$\pi_* \mathcal{O}_{X \, |V} \cong \bigoplus_{\chi \in Irr(V)} \mathcal{O}_{Y|V} \cdot u_\chi^V$$

so the cocycles of the line bundles V_χ^{-1} are given by $\rho_{21} = \frac{u_\chi^{V_2}}{u_\chi^{V_1}}$ on $V_1 \cap V_2$, which are by construction of $V(\bigoplus_{\chi \in Irr(G)} L_\chi)$ the cocycles of L_χ.

We have proved $\{L_\chi\}_\chi$ and $\{D_g\}_{g \in G}$ are the building data of $\pi_{|X}: X \to Y$.

Regardless the uniqueness, we discuss it below. ▨

Rem: We observe that the Pardini construction of an abelian cover from a set of line bundles $\{L_\chi\}_\chi$ and divisors $\{D_g\}_g$ works in general, without any other assumption on $\{L_\chi\}_\chi$ and $\{D_g\}_g$. Thus, one can study the properties of $\{L_\chi\}_\chi$ and $\{D_g\}_g$ for which the obtained Galois cov. is connected, smooth, normal, ecc..

Def Given a variety Y, a finite group G, and a set of line bundles $\{L_\chi\}_\chi$ and divisors $\{D_g\}_{g\in G}$ on Y satisfying eq. (*) above, then we define the standard abelian cover $\pi: X \to Y$, the cover constructed in the proof of Pardini Ex. Thm.
In this case, we refer to $\{L_\chi\}_\chi, \{D_g\}$ as to the building data of the cover, and to $D = \sum_{g\in G} D_g$ as the branch locus also if X is not normal.

Lemma Any normal abelian cover is standard. Instead, not all standard abelian covers are normal.

proof Assume to have a Galois covering $\pi: X \to Y$ with group G, X normal, and Building Data $\{L_\chi\}_\chi, \{D_g\}_g$. Let $X' \xrightarrow{\pi'} Y$ be the abelian covering with group G constructed as above from $\{L_\chi\}_\chi, \{D_g\}_g$, where we choose as sections σ_h of $\Delta_h \subseteq D_h$ exactly the invariant function $\sigma_{h|V} := y_1$ on Y.

We remind that $s_\chi := \{(\pi^{-1}(V), \tau_V^\chi)\}_{V \subseteq Y}$ are global sections of $\pi^* L_\chi$, $\chi \in G^*$. Then an isomorphism among $X \xrightarrow{\pi} Y$ and $X' \xrightarrow{\pi'} Y$ is given by

$$\begin{aligned} X &\xrightarrow{\ \Psi\ } X' \\ p &\longmapsto (\pi(p), s_\chi(p), \chi \in G^* \setminus 1_G) \end{aligned}$$

$$\begin{array}{ccc} X & \xrightarrow{\ \Psi\ } & X' \\ & {\scriptstyle \pi}\searrow & \downarrow {\scriptstyle \pi'} \\ & & Y \end{array}$$

∎

Prop (Uniqueness of Pardini Existence Thm.)

Let $\pi: X \to Y$ and $\pi': X' \to Y$ be two standard abelian covers with group G and same building data $\{L_\chi\}_\chi$, $\{D_g\}_g$. If Y is (complete?) then π and π' are isomorphic as Galois covers

proof Let us prove it first in the case Y is complete. X and X' are standard ab. covers with same $\{L_\chi\}_\chi$ and $\{D_g\}_{g\in G}$, so they are defined choosing

$$\sigma_g \in H^0(Y, \mathcal{O}_Y(D_g)), \ (\sigma_g) = D_g$$
$$\sigma'_g \in H^0(Y, \mathcal{O}_Y(D_g)), \ (\sigma'_g) = D_g$$

Thus, given $V \subseteq Y$ open subset trivializing simultaneously L_χ, $\chi \in G^*$, we have

$$X \cap \pi^{-1}(V) = \left\{ u_\chi u_{\chi'} = \left(\prod_{r_\chi^g + r_{\chi'}^g \geq |g|} \sigma_g \right) \cdot u_{\chi\chi'} \right\}$$

$$X' \cap (\pi')^{-1}(V) = \left\{ \tilde{u}_\chi \tilde{u}_{\chi'} = \left(\prod_{r_\chi^g + r_{\chi'}^g \geq |g|} \sigma'_g \right) \tilde{u}_{\chi\chi'} \right\}$$

We need to find an isomorphism of Galois covers between X and X'!

Since $(\sigma_g) = (\sigma'_g) = D_g \implies \frac{\sigma'_g}{\sigma_g} \in \mathcal{O}_Y^*(Y)$ is a global nowhere zero global holomorphic function of Y.

Since Y is complete, then $\frac{\sigma'_g}{\sigma_g} = \text{const} \neq 0$, and we can consider for any $\chi \in G^*$ the first $|\chi|$-root of $\left(\frac{\sigma'_g}{\sigma_g}\right)^{r_\chi^g}$:

$$\gamma_\chi := \prod_{g\in G} \sqrt[|\chi|]{\left(\frac{\sigma'_g}{\sigma_g}\right)^{r_\chi^g}}$$

Then an isomorphism is

$$X \xrightarrow{\Psi} X'$$

$$\text{on } \pi^{-1}(V) \quad (p, (u_\chi : \chi \in G^*\setminus 1)) \longmapsto (p, (\lambda_\chi u_\chi : \chi \in G^*\setminus 1))$$

Let us prove it is well defined, namely $\Psi(p) \in X'$:

$\tilde{u}_\chi = \lambda_\chi u_\chi$ and

$$\tilde{u}_\chi \tilde{u}_{\chi'} = \lambda_\chi \lambda_{\chi'} u_\chi u_{\chi'} = \lambda_\chi \lambda_{\chi'} \cdot \prod_{r_\chi^g + r_{\chi'}^g \geq |g|} \sigma_g \cdot u_{\chi\chi'}$$

$$= \left(\prod_{r_\chi^g + r_{\chi'}^g \geq |g|} \frac{\sigma_g'}{\sigma_g} \right) \cdot \prod_{r_\chi^g + r_{\chi'}^g \geq |g|} \sigma_g \cdot \lambda_{\chi\chi'} u_{\chi\chi'} = \prod_{r_\chi^g + r_{\chi'}^g \geq |g|} \sigma_g' \cdot \tilde{u}_{\chi\chi'}$$

✓

$$\lambda_\chi \lambda_{\chi'} = \prod_{g \in G} \sqrt[|g|]{\left(\frac{\sigma_g'}{\sigma_g}\right)^{r_\chi^g + r_{\chi'}^g}} = \begin{cases} \prod_{g \in G} \sqrt[|g|]{\left(\frac{\sigma_g'}{\sigma_g}\right)^{r_{\chi\chi'}^g}} & \text{if } r_\chi^g + r_{\chi'}^g < |g| \\ \prod_{g \in G} \frac{\sigma_g'}{\sigma_g} \cdot \sqrt{\left(\frac{\sigma_g'}{\sigma_g}\right)^{r_{\chi\chi'}^g}} & \text{if } r_\chi^g + r_{\chi'}^g \geq |g| \end{cases} = \begin{cases} \lambda_{\chi\chi'} & \text{if } r_\chi^g + r_{\chi'}^g < |g| \\ \prod_{g \in G} \frac{\sigma_g'}{\sigma_g} \cdot \lambda_{\chi\chi'} & \text{oth.} \end{cases}$$

Assume now Y is not complete. Then, up to restrict the trivializing open set $V \subseteq Y$, Y can be supposed to be simply connected, so we can define the $|X|$-root $\lambda_{\chi|V}$ as above on V. Then the proof follows with a similar argument as above (probably there could be a problem on multiplying u_χ by $\lambda_{\chi|V}$. Indeed, if we change the open set V, then $\lambda_{\chi|V}$ and $\lambda_{\chi|V'}$ could not be compatible and Ψ is not well-defined)

▨

Remark The previous Prop gives the uniqueness part of the Pardini Ex. Thm.

Indeed, let $\pi: X \to Y$ and $\pi': X' \to Y$ be two normal abelian covers with group G and same building data $\{\mathcal{L}_\chi\}_{\chi\in G^*}$, $\{D_g\}_{g\in G}$.

Then $\pi: X \to Y$ is isom. with its standard ab. cover $\pi_{st}: X_{st} \to Y$, which has building data $\{\mathcal{L}_\chi\}_\chi$, $\{D_g\}_g$ via the sections σ_g of D_g,

and the same holds for $\pi': X' \to Y$ and $\pi'_{st}: X'_{st} \to Y$, whose isomorphism is given by sections σ'_g of D_g.

However, π_{st} and π'_{st} have the same $\{\mathcal{L}_\chi\}_\chi$ and $\{D_g\}_g$, so by the prev. Prop. are isomorphic by an isomorphism $\phi_{st}: X_{st} \to X'_{st}$. Thus, $\phi := (\Psi')^{-1} \circ \phi_{st} \circ \Psi : X \to X'$ is an iso:

$$\begin{array}{ccc} X & \xrightarrow{\ \phi\ } & X' \\ \Psi \downarrow \wr & & \wr \downarrow \Psi' \\ X_{st} & \xrightarrow[\phi_{st}]{\ \sim\ } & X'_{st} \end{array}$$

Remark The number r_χ^h is very easy to compute for an elementary abelian p-group $G \cong (\mathbb{Z}/p)^k$. Indeed, all the elements of G and G^* have the same order p, so $0 \leq r_\chi^h \leq p-1$ is the unique integer s.t. $\chi(h) = e^{\frac{2\pi i}{p} r_\chi^h}$.

For instance, for $G = \langle e_1, e_2 \rangle \cong \mathbb{Z}_5^2$, $G^* = \langle \varepsilon_1, \varepsilon_2 \rangle$, $r^{2e_1+e_2}_{\varepsilon_1+\varepsilon_2} = 3$.

Example 1. $Y = \mathbb{P}^2(y_0, y_1, y_2)$, $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle e_1, e_2 \rangle$

$D_{e_1} := (y_0 = 0)$, $D_{e_2} = (y_1 = 0)$, $D_{e_1+e_2} = (y_2 = 0)$

Can we construct a $\mathbb{Z}_2 \times \mathbb{Z}_2$ - cover of $\mathbb{P}^2$ with branch locus $D_{e_1} + D_{e_2} + D_{e_1+e_2}$?

We need to determine, if they exists, line bundles $\{L_\chi\}_\chi$ such that Pardini equations hold.
We can try to find them using Pardini Equations:

$$2L_{\varepsilon_1} = \underset{\substack{\| \\ L_0 = 0}}{L_{2\varepsilon_1}} + \sum_{\substack{g:\, r_{\varepsilon_1}^g = 1 \\ \Updownarrow \\ g \notin \ker(\varepsilon_1)}} D_g = D_{e_1} + D_{e_1+e_2} \equiv 2H$$

$\Rightarrow$ we need to choose $L_{\varepsilon_1} \equiv \mathcal{O}_{\mathbb{P}^2}(H)$.
($\Downarrow$ $\mathrm{Pic}(\mathbb{P}^2)$ has not torsion)

Similarly, $2L_{\varepsilon_2} \equiv 0 + \sum_{g \notin \ker(\varepsilon_2)} D_g = D_{e_2} + D_{e_1+e_2} = 2H$

$$\Rightarrow L_{\varepsilon_2} \equiv \mathcal{O}_{\mathbb{P}^2}(H)$$

Instead, $2L_{\varepsilon_1+\varepsilon_2} \equiv 0 + \sum_{g \notin \ker(\varepsilon_1+\varepsilon_2)} D_g = D_{e_1} + D_{e_2} = 2H$

$$\Rightarrow L_{\varepsilon_1+\varepsilon_2} \equiv \mathcal{O}_{\mathbb{P}^2}(H)$$

$2H \equiv L_{\varepsilon_1} + L_{\varepsilon_2} = L_{\varepsilon_1+\varepsilon_2} + \sum_{g \notin \ker(\varepsilon_1) \cap \ker(\varepsilon_2)} D_g = L_{\varepsilon_1+\varepsilon_2} + D_{e_1+e_2} \equiv 2H$ ✓

$2H \equiv L_{\varepsilon_1} + L_{\varepsilon_1+\varepsilon_2} = L_{\varepsilon_2} + D_{e_1} \equiv 2H$ ✓ $\quad 2H \equiv L_{\varepsilon_2} + L_{\varepsilon_1+\varepsilon_2} = L_{\varepsilon_1} + D_{e_2} \equiv 2H$ ✓

Thus, $L_{\varepsilon_1} = L_{\varepsilon_2} = L_{\varepsilon_1+\varepsilon_2} \equiv \mathcal{O}_Y(H)$ and $D_{e_1} \equiv D_{e_2} \equiv D_{e_3} \equiv H$ satisfy Pardini Equations $\Rightarrow \exists!$ Galois Covering of $\mathbb{P}^2$ with group $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ and b. data $\{L_\chi\}_\chi, \{D_g\}_g$.
This abelian cover is that of Example 4 of the 1st Lecture